

# Non-supersymmetric asymptotically $\text{AdS}_5 \times S^5$ smooth geometries

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**ABSTRACT:** We find soliton solutions in five-dimensional gauged supergravity, where a circle degenerates smoothly in the core of the geometry. In the family of solutions we consider, we find no completely smooth supersymmetric solutions, but we find discrete families of non-supersymmetric solitons. We discuss the relation to previous studies of the asymptotically flat case. We also consider gauged supergravities in four and seven dimensions, but fail to find any smooth solutions.

**KEYWORDS:** AdS-CFT Correspondence, Gauge-gravity correspondence, Black Holes in String Theory.

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## Contents

<b>1. Introduction</b>	<b>1</b>
<b>2. Equal charges and equal angular momenta</b>	<b>3</b>
<b>3. Unequal charges and equal angular momenta</b>	<b>7</b>
<b>4. Equal charges and unequal angular momenta</b>	<b>10</b>
<b>5. Asymptotically Poincaré solutions</b>	<b>14</b>
<b>6. Asymptotically flat limit</b>	<b>16</b>
<b>7. Conclusions</b>	<b>18</b>

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## 1. Introduction

There has recently been considerable interest in smooth supergravity solutions, and their relation to the states of dual conformal field theory (CFT) descriptions. This was initiated in studies of the D1-D5 system [1, 2], where smooth geometries whose near-horizon limit is global  $\text{AdS}_3 \times S^3$  were found. This work has been extended in a number of interesting ways, as will be reviewed below. More recently, 1/2 BPS smooth asymptotically  $\text{AdS}_5 \times S^5$  solutions were found in [3]. Another class of asymptotically  $\text{AdS}_5$  solutions was recently found in [4–6]. As in [1, 2], these latter solutions involve a degenerating circle, so they are more closely akin to the D1-D5 system than to the bubbling AdS solutions of [3]. The aim of the present paper is to construct new examples of this type in gauged supergravity, employing the approaches used in the study of the D1-D5 system (the analysis will be closely based on the approach of [7]).

To begin with, let us briefly review the previous work in the D1-D5 system. In [1, 2], special smooth solutions were found in the family of asymptotically flat supersymmetric geometries [8] describing a D1-D5 system wrapped on a circle, with angular momentum in the transverse space. The near-horizon limit of these metrics was global  $\text{AdS}_3 \times S^3$ . By studying the near-horizon limit, they were identified with Ramond-Ramond ground states of the dual CFT. They were generalised in [9–11] to obtain a family depending on arbitrary functions, corresponding to more general Ramond-Ramond ground states. These were interpreted in [11] as Kaluza-Klein (KK) monopole supertubes; the five-dimensional geometry has a KK monopole wrapped on a contractible circle.

This construction was then generalised to include momentum along the D1-D5 string in [12–14, 7, 15, 16]. The resulting five-dimensional geometries are smooth up to some

orbifold singularities, and there are again large families of solitons. The CFT interpretation of the most general solutions in this class has not yet been understood. In [7], the first non-supersymmetric solitons were found, by returning to the general family of metrics in [8] and considering the restrictions on the parameters necessary to ensure smoothness. States of the dual CFT corresponding to the  $\text{AdS}_3 \times S^3$  geometries obtained in the near-horizon limit were also identified. It is really remarkable that such non-supersymmetric solitons exist, and further exploration of them may provide important insights into the relation between CFT states and geometry.

In this paper, we will find new non-supersymmetric soliton solutions by applying the approach used in [7] to gauged supergravity. That is, we consider the families of five-dimensional asymptotically  $\text{AdS}_5$  charged rotating black holes found in [17, 18, 5, 6], and ask what restrictions on the parameters are implied by requiring that the geometry closes off smoothly in the interior with a degenerating circle. This problem was already considered for the supersymmetric limit of the solutions in [4–6], where special cases that give smooth metrics were identified. We generalise this by considering the general non-supersymmetric solutions of [17, 18, 5, 6], and we consider the conditions necessary to ensure the matter fields are also smooth. The latter conditions turn out not to be satisfied by any of the supersymmetric solutions in [4–6].<sup>1</sup> That is, once we impose these gauge field smoothness conditions, we have no smooth supersymmetric solitons. There are discrete families of smooth non-supersymmetric solitons which satisfy all these conditions.<sup>2</sup>

The bulk of the paper is taken up with the analysis of the different families of solutions from [17, 18, 5, 6]. In the next section, we start by considering the simplest case, a charged rotating black hole in minimal gauged supergravity with three equal charges and two equal angular momenta. We then extend this to unequal charges, in section 3, or to unequal angular momenta, in section 4. (Solutions with both unequal charges and unequal angular momenta are not yet available in the literature.) In section 5, we obtain solutions which are asymptotically  $\text{AdS}_5$  in Poincaré coordinates by taking a large mass limit of the previous solutions. The same geometries can be obtained by double analytic continuation of charged black holes. In section 6, we consider the limit of vanishing cosmological constant, and discuss the relation to the asymptotically flat solutions of [7].

The extension from asymptotically flat to asymptotically  $\text{AdS}_5$  boundary conditions is interesting for two main reasons. It offers the possibility of finding some dual description of these solutions in a four-dimensional CFT, living on the boundary of  $\text{AdS}_5$ . (However, the fact that all the solitons we find are non-supersymmetric makes it more difficult to obtain a CFT description.) It is also interesting because black holes in  $\text{AdS}$  are better behaved than black holes in asymptotically flat space. If we think of these geometries as

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<sup>1</sup>There is a component of the gauge potential along the degenerating direction which does not vanish where the circle degenerates. This corresponds to twisting the degenerating circle over a circle in the  $S^5$  by a non-integer amount, in the description of this gauged supergravity as arising from Kaluza-Klein reduction of IIB on an  $S^5$ .

<sup>2</sup>There are other asymptotically locally  $\text{AdS}$  solitons which have been constructed by different methods: The  $\text{AdS}$  soliton of [19] asymptotically approaches a  $\mathbb{Z}$  quotient of  $\text{AdS}$  in Poincare coordinates, and other asymptotically  $\text{AdS}/\mathbb{Z}_p$  solitons were recently constructed in [20, 21].

describing microstates of black holes, following the proposal of Mathur and collaborators [9, 10] (see also the review [22]), then the asymptotically AdS<sub>5</sub> case may make it easier to make quantitative comparisons. In section 7, we offer some concluding comments on these and other issues and briefly discuss the extension of the approach used here to gauged supergravities in other dimensions. We consider the four and seven dimensional theories, and argue that the known families of charged rotating black hole solutions do not contain any solitons as special cases.

## 2. Equal charges and equal angular momenta

We start from the solution of [17], describing a charged rotating black hole in minimal five-dimensional gauged supergravity. We work in the parametrisation introduced in [23], but write the cosmological constant as  $\lambda = -g^2$ . The metric is

$$ds^2 = -\frac{r^2 W}{4b^2} dt^2 + \frac{1}{W} dr^2 + \frac{r^2}{4} (\sigma_1^2 + \sigma_2^2) + b^2 (\sigma_3 + f dt)^2, \quad (2.1)$$

$$A = \frac{\sqrt{3}q}{r^2} \left[ dt - \frac{1}{2} j \sigma_3 \right], \quad (2.2)$$

where

$$b^2 = \frac{1}{4r^4} [r^6 - j^2 q^2 + 2j^2 p r^2], \quad (2.3)$$

$$f = -\frac{j}{2b^2} \left[ \frac{2p - q}{r^2} - \frac{q^2}{r^4} \right], \quad (2.4)$$

$$W = 1 + g^2 r^2 + \frac{1}{r^2} [2j^2 g^2 p - 2(p - q)] + \frac{1}{r^4} [q^2 + j^2 (2p - g^2 q^2)], \quad (2.5)$$

and the  $\sigma_i$  are the left-invariant one-forms on  $S^3$ ,

$$\sigma_1 = \cos \psi d\theta + \sin \psi \sin \theta d\phi, \quad (2.6)$$

$$\sigma_2 = -\sin \psi d\theta + \cos \psi \sin \theta d\phi, \quad (2.7)$$

$$\sigma_3 = d\psi + \cos \theta d\phi. \quad (2.8)$$

The determinant of the metric is  $g = -r^6/16$ , so the only potential singularities are at  $W = 0$ ,  $b = 0$  (where  $f$  diverges) and  $r = 0$ . Since the determinant goes like  $r^6$ , the singularity at  $r = 0$  is a real singularity, unlike in the analysis of the D1-D5 case in [7], where it could be removed by a coordinate transformation. The determinant of the metric on a surface of constant  $r$  is  $g|_{r=r_0} = -r_0^6 W/16$ , so the singularity at  $W = 0$  corresponds either to a horizon or to a degeneration in the spatial metric. The determinant of the spatial metric is  $g|_{r=r_0, t=t_0} = r^4 b^2/4$ , so there is a degeneration in the spatial metric if  $b = 0$  as well.

Thus, to have a smooth solution with a circle degenerating in the interior, there needs to be some  $r = r_0 > 0$  which is, firstly, the largest root of  $W$ , so  $W(r_0) = 0$  and  $W(r) > 0$  for  $r > r_0$ . Secondly, it must be the largest root of  $b$ , so  $b^2(r_0) = 0$  and  $b^2(r) > 0$  for  $r > r_0$ . Thirdly, there must be no  $dt d\psi$  cross terms at  $r = r_0$ , as they would prevent

us from constructing a smooth Cartesian coordinate system near this origin; this requires  $b^2 f(r_0) = 0$ , or equivalently that  $f$  be finite at  $r = r_0$ . This final condition is the easiest one to solve, and gives us

$$r_0^2 = \frac{q^2}{2p - q}. \tag{2.9}$$

Requiring that  $W(r_0) = 0$  and  $b^2(r_0) = 0$  then imposes conditions on the parameters; these conditions are both satisfied if

$$j^2 = -\frac{q^3}{(2p - q)^2}, \tag{2.10}$$

which requires  $q < 0$ .<sup>3</sup> The BPS bound is then  $p \geq 0$ . This  $r_0$  is automatically the largest root of  $b^2$ . Requiring it to be the largest root of  $W$  implies

$$w \equiv g^2 q^3 + 2p(2p - q) < 0. \tag{2.11}$$

With this choice of parameters, as  $r \rightarrow r_0$ , the proper size of the  $\psi$  circle goes to zero. An important difference relative to the case studied in [7] is that the period of  $\psi$  is already fixed; to have an asymptotically AdS<sub>5</sub> spacetime,  $\Delta\psi = 4\pi$ . More generally,  $\phi, \psi, \theta$  could define a smooth quotient of  $S^3$  asymptotically, which would permit  $\Delta\psi = 4\pi/k$  for integer  $k$ . (Our main interest will be in the asymptotically AdS<sub>5</sub> case  $k = 1$ .) Because the periodicity of  $\psi$  is already fixed, the condition that  $r = r_0$  is a smooth origin in the plane is going to impose an additional restriction on the parameters. Near  $r = r_0$ , the relevant part of the metric is

$$ds^2 = \dots + \frac{(2p - q)q^2 dx^2}{(3q - 2p)w} + \frac{3q - 2p}{4q} x^2 d\psi^2, \tag{2.12}$$

where  $x^2 \equiv r^2 - r_0^2$ , and  $w$  is given in (2.11). We must then have  $3q - 2p < 0$  and  $w < 0$  for the metric to have the correct signature. The regularity condition that  $r = r_0$  be a smooth origin then requires

$$\frac{(2p - 3q)^2}{(2p - q)q^3} (g^2 q^3 + 2p(2p - q)) = k^2. \tag{2.13}$$

When the two conditions (2.10) and (2.13) are satisfied, the metric is completely smooth, with a smooth origin at  $r = r_0$ . Since  $W > 0$  for  $r > r_0$ , the surfaces of constant  $r$  are timelike; hence, there are timelike curves that move to  $r \rightarrow \infty$  from any point in the spacetime, showing that there are no event horizons in this spacetime. Since  $g^{tt} = -4b^2/(r^2 W) < 0$  for  $r \geq r_0$ ,  $t$  is a global time function for our soliton, and the metric is stably causal (and hence free of closed timelike curves).

So far we have imposed two conditions on the parameters, (2.10) and (2.13). This leaves one free parameter in this class of solutions. However, there is another regularity condition that we need to impose: regularity of the matter fields. The gauge potential near  $r = r_0$  is

$$A = \frac{\sqrt{3}(2p - q)}{q} dt - \frac{\sqrt{-3q}}{2} \sigma_3, \tag{2.14}$$

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<sup>3</sup>Note that the situation is not symmetric under  $q \rightarrow -q$  because of the presence of a Chern-Simons term in the gauged supergravity Lagrangian.

so  $A_\psi$  does not vanish near  $r = r_0$ . The holonomy around this direction,

$$\oint_{S^1} A = -\frac{\sqrt{-3q}}{2} \Delta\psi, \tag{2.15}$$

is a gauge invariant quantity. This holonomy seems to provide an obstruction to making  $r = r_0$  a regular origin; If  $r = r_0$  was a regular origin this circle would be a contractible cycle, and one would expect

$$\oint_{S^1} A = \int_{D^2} F \rightarrow 0 \tag{2.16}$$

as the circle contracts. However, the gauge potential  $A$  comes from a diagonal  $U(1) \subset U(1)^3 \subset SO(6)$ . Since the gauge group is compact, the holonomy takes values in a circle, and can be shifted by an integer. Thus, regularity only requires  $\oint_{S^1} A \rightarrow 0$  modulo an appropriate period.

The issue is perhaps most easily understood by recalling that from the ten-dimensional perspective, these are Kaluza-Klein gauge fields, and the gauge field  $A$  comes from writing a Kaluza-Klein ansatz [24]

$$ds_{10}^2 = ds_5^2 + \frac{1}{g^2} \sum_i \left[ d\mu_i^2 + \mu_i^2 \left( d\phi_i + \frac{g}{\sqrt{3}} A \right)^2 \right], \tag{2.17}$$

where  $\mu_i, \phi_i$   $i = 1, \dots, 3$  are coordinates on an  $S^5$ , so  $\sum_i \mu_i^2 = 1$  and  $\Delta\phi_i = 2\pi$ . There are globally well-defined large coordinate transformations in this metric,  $\phi \rightarrow \tilde{\phi} = \phi + \frac{km}{2}\psi$  for integer  $m$ . These coordinate transformations preserve the periodicity of the coordinates; that is,  $\tilde{\phi} \sim \tilde{\phi} + 2\pi$  and  $\psi \sim \psi + 4\pi/k$  generate the same identifications as  $\phi \sim \phi + 2\pi$  and  $\psi \sim \psi + 4\pi/k$ . From the Kaluza-Klein perspective, these coordinate transformations correspond to large gauge transformations  $A \rightarrow A + \frac{\sqrt{3}km}{2g} d\psi$ . They can therefore be used to remove the non-zero  $A_\psi$  near  $r = r_0$  if

$$\sqrt{-q} = \frac{km}{g}. \tag{2.18}$$

That is, the condition on the holonomy is  $\oint_{S^1} A \rightarrow 0 \pmod{2\pi\sqrt{3}/g}$ , which is satisfied if (2.18) is. This condition tells us that the twisting of the degenerating circle on the  $S^5$  is quantized. Thus  $m$  is analogous to the integers specifying the smooth solitons in [7].

Thus, for a given asymptotic boundary condition, there is a one-integer parameter family of smooth solitons. In particular, for the asymptotically  $AdS_5 \times S^5$  solutions,  $k = 1$ , and the solitons are labelled by  $m = 1, 2, \dots$ . In table 1, the values of the parameters for the first few solitons are listed. Note that there is no supersymmetric soliton; that is, there is no case with  $p = 0$ .<sup>4</sup>

Since our parametrisation is not particularly physical, it is more useful to give the asymptotic charges. It is clear from the form of the gauge potential that the conserved

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<sup>4</sup>In particular, the equal-charge case of the supersymmetric solutions in [4] does not satisfy (2.18). This was noted indirectly in [4], where it was observed that the correct periodicity conditions for the gravitini could not be satisfied for the solutions considered there. The relation to [4] will be discussed in more detail in the next section.

$m$	$p$	$q$	$j$	$M$
1	0.281	-1	0.640	3.108
2	3.063	-4	0.790	18.143
3	11.351	-9	0.852	54.416
4	28.143	-16	0.885	121.337

**Table 1:** The values of the parameters and thermodynamic mass for the first few solitons, for  $k = 1$ . All quantities are quoted in units in which  $g = 1$ .

charge is  $q$ , up to a normalisation factor. The angular momentum  $J$  can be calculated using the Komar integral technique and is given by

$$J = \frac{\pi}{4} j(2p - q) = \frac{\pi}{4g^3} k^3 m^3, \tag{2.19}$$

so the angular momentum has a simple expression in terms of  $m$ . A thermodynamic mass was obtained for this general family of black holes in [23]:

$$M = \frac{\pi}{4} (3(p - q) + pj^2 g^2). \tag{2.20}$$

This does not have such a simple form in terms of the basic parameters, but its value for the first few solitons is listed in table 1.

An interesting difference between these solitons and the asymptotically flat ones discussed in [7] is that, as suggested in [7], they do not have an ergoregion. That is, the Killing vector  $V = \partial_t$  is everywhere timelike or null. Writing  $r^2 = r_0^2 + z^2$ , its norm is

$$g_{\mu\nu} V^\mu V^\nu = -\frac{z^2}{4b^2(r_0^2 + z^2)^4} P_{10}(z), \tag{2.21}$$

where  $P_{10}(z)$  is a tenth-order even polynomial in  $z$ . The coefficients in this polynomial are all positive when  $r_0^2$  is given by (2.9) and  $j^2$  is given by (2.10), so the norm is non-positive for  $z^2 \geq 0$ , that is, for  $r^2 \geq r_0^2$ . The absence of an ergoregion suggests that these geometries stand a better chance of being stable than the solitons in [7]. It would clearly be interesting to explore this issue in more detail.

To close this section, we make a remark about spin structures on these solutions for the case  $k = 1$ .<sup>5</sup> The surfaces of constant  $r$  have topology  $\mathbb{R} \times S^3$ , so they have a unique spin structure. The metric on  $S^3$  can be written as

$$\begin{aligned} ds_{S^3}^2 &= \frac{r^2}{4} (d\theta^2 + \sin^2 \theta d\phi^2) + \frac{b^2}{4} (d\psi + \cos \theta d\phi)^2 \\ &= r^2 (d\vartheta^2 + \sin^2 \vartheta \cos^2 \vartheta (d\phi_1 - d\phi_2)^2) + b^2 (\cos^2 \vartheta d\phi_1 + \sin^2 \vartheta d\phi_2)^2, \end{aligned} \tag{2.22}$$

where  $\theta = 2\vartheta$ ,  $\psi = \phi_1 + \phi_2$ ,  $\phi = \phi_1 - \phi_2$ . Now the  $\phi_{1,2}$  circles are contractible (they shrink to zero at  $\theta = 0, \pi$  respectively), so the spin structure must have antiperiodic fermions around these circles. This implies the fermions are periodic around the  $\psi$  circle. But this

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<sup>5</sup>This issue is discussed in detail in the revised version of [4].

does not define a consistent spin structure on the full spacetime, as the  $\psi$  circle is also contractible: it degenerates at  $r = r_0$ . Thus, the five-dimensional solutions are not spin manifolds.

However, this problem is easily resolved in the ten-dimensional geometry. The  $\psi$  circle at fixed position on the  $S^5$  has a periodic spin structure by the above argument. But the circle which degenerates smoothly in the interior is the  $\psi$  circle at fixed values of  $\tilde{\phi}_i = \phi_i + \frac{m}{2}\psi$ ; that is, it corresponds to going around the  $\psi$  circle while going around each of the  $\phi_i$  circles  $m$  times. Since the fermions must also be antiperiodic around the  $\phi_i$  circles, they will be antiperiodic around the degenerating cycle, as required, for odd values of  $m$ . Thus, for odd  $m$ , the ten-dimensional spacetime has a unique well-defined spin structure. From the five-dimensional point of view, the presence of the Kaluza-Klein gauge field allows us to define a spin<sup>c</sup> structure on the spacetime even though it does not have a spin structure. That is, the fermions taking values in this spin bundle on the ten dimensional spacetime Kaluza-Klein reduce to give us spinor fields in five dimensions charged under the gauge field.

### 3. Unequal charges and equal angular momenta

The equal-charge solution studied above was generalised to unequal charges in [18]. This more general family also contains smooth solitons, but there are still no supersymmetric cases, contrary to the claims in [4]. The generalisation means that the solitons will be labelled by three integer parameters determining the charges, replacing the single parameter  $m$  in the above solutions.

We start from the solutions of [18] in the form introduced in [23], where the metric and gauge fields are

$$ds^2 = -\frac{RY}{f_1}dt^2 + \frac{R\bar{\rho}^2}{Y}d\bar{\rho}^2 + \frac{1}{4}R(\sigma_1^2 + \sigma_2^2) + \frac{f_1}{4R^2} \left( \sigma_3 - 2\frac{f_2}{f_1}dt \right)^2, \quad (3.1)$$

and

$$A^i = \frac{r_j r_k - \Gamma r_i}{\bar{\rho}^2 \tilde{H}_i} dt + \frac{L r_i}{2\bar{\rho}^2 \tilde{H}_i} \sigma_3, \quad (3.2)$$

with

$$f_1 = R^3 + L^2 \left( \bar{\rho}^2 - \frac{1}{3} \sum_i r_i^2 \right), \quad (3.3)$$

$$f_2 = L \left( \Gamma \bar{\rho}^2 - \frac{1}{3} \Gamma \sum_i r_i^2 + r_1 r_2 r_3 \right), \quad (3.4)$$

$$Y = -\lambda R^3 + \bar{\rho}^4 + \left( \frac{1}{3} \sum_i r_i^2 - \lambda L^2 - \Gamma^2 \right) \bar{\rho}^2 + \frac{1}{3} \Gamma^2 \sum_i r_i^2 - 2\Gamma r_1 r_2 r_3 + \frac{1}{3} \lambda L^2 \sum_i r_i^2 + L^2 + \left[ \frac{5}{18} \left( \sum_i r_i^2 \right)^2 - \frac{1}{2} \sum_i r_i^4 \right], \quad (3.5)$$



and

$$R = \bar{\rho}^2 \left( \prod_i^3 \tilde{H}_i \right)^{\frac{1}{3}}, \quad \tilde{H}_i = 1 + \frac{2r_i^2 - r_j^2 - r_k^2}{3\bar{\rho}^2}. \quad (3.6)$$

Because the charges are unequal, this solution also involves non-trivial scalar fields  $\varphi_1, \varphi_2$ , given by

$$e^{\frac{2}{\sqrt{6}}\varphi_1} = X_3, \quad e^{\sqrt{2}\varphi_2} = \frac{X_2}{X_1}, \quad X_i = \frac{R}{\bar{\rho}^2 \tilde{H}_i}. \quad (3.7)$$

The parametrisation of the solutions above was introduced in [23] to demonstrate the uniqueness of the solutions. However, it is not the most physically transparent or convenient parametrisation, so before searching for smooth solitons, we would like to introduce a more convenient parametrisation and a minor coordinate transformation. Introduce

$$\ell_1^2 = \frac{r_1 r_2 r_3}{\Gamma}, \quad \ell_2^6 = L^2 \ell_1^2, \quad q_i = \frac{r_i^2}{\ell_1^2} - 1, \quad (3.8)$$

set as usual  $\lambda = -g^2$ , and shift the radial coordinate,

$$\bar{\rho}^2 = \rho^2 + \frac{\ell_1^2}{3}(q_1 + q_2 + q_3). \quad (3.9)$$

The powers are chosen so that  $\ell_2$  and  $\ell_1$  have length dimensions, and  $q_i$  are dimensionless. In this parametrisation,

$$ds^2 = -\frac{RY}{f_1} dt^2 + \frac{R\rho^2}{Y} d\rho^2 + \frac{1}{4}R(\sigma_1^2 + \sigma_2^2) + \frac{f_1}{4R^2} \left( \sigma_3 - 2\frac{f_2}{f_1} dt \right)^2, \quad (3.10)$$

with

$$f_1 = R^3 + \frac{\ell_2^6}{\ell_1^2} \rho^2 - \ell_2^6, \quad (3.11)$$

$$f_2 = \ell_2^3 \sqrt{q_1 + 1} \sqrt{q_2 + 1} \sqrt{q_3 + 1} \rho^2, \quad (3.12)$$

$$Y - g^2 f_1 = \rho^4 - \rho^2 \ell_1^2 [q_1 q_2 + q_1 q_3 + q_2 q_3 + q_1 q_2 q_3] + \frac{\ell_2^6}{\ell_1^2} - \ell_1^4 q_1 q_2 q_3, \quad (3.13)$$

and  $R^3 = \bar{H}_1 \bar{H}_2 \bar{H}_3$ , where

$$\bar{H}_i = \bar{\rho}^2 \tilde{H}_i = \rho^2 + \ell_1^2 q_i, \quad (3.14)$$

and the gauge field becomes

$$A^i = -q_i \sqrt{q_j + 1} \sqrt{q_k + 1} \frac{\ell_1^2}{(\rho^2 + \ell_1^2 q_i)} dt + \sqrt{q_i + 1} \frac{\ell_2^3}{2(\rho^2 + \ell_1^2 q_i)} \sigma_3. \quad (3.15)$$

Thus, the parameters  $q_i$  are proportional to the gauge charges.

Let us now investigate the conditions for a smooth origin. As in the previous case, there will be a smooth origin where the  $\psi$  circle degenerates at  $\rho^2 = \rho_0^2$  if both the determinant of the metric on surfaces of constant  $r$  vanishes there and the determinant of the metric on surfaces of constant  $r$  and  $t$  vanishes there. This requires that  $\rho^2 = \rho_0^2$  is the largest root of  $f_1$  and  $Y$ . This must also be a root of  $f_2$  to eliminate cross terms, and we must

have  $\bar{H}_i(\rho_0) > 0$  to ensure there are no singularities elsewhere. Requiring that  $\rho_0$  is a root of  $f_2$  determines  $\rho_0^2 = 0$ , so  $\bar{H}_i(\rho_0) > 0$  implies  $\ell_1^2 q_i > 0$ . As in the equal charge case, this corresponds to negative gauge charges. Requiring  $f_1(\rho = 0) = 0$  implies

$$\ell_2^6 = R^3(\rho = 0) = \ell_1^6 q_1 q_2 q_3, \tag{3.16}$$

and it is easy to see that  $Y(\rho = 0) = 0$  as well. This is automatically the largest root of  $f_1$ . It will be the largest root of  $Y$  as well if  $g^2 \ell_1^2 > 1$ .

We now have the proper size of the  $\psi$  circle going to zero at  $\rho = 0$ . As in the previous section, the period of the  $\psi$  circle is already fixed by imposing regularity of the metric on the spheres of constant  $\rho$  and  $t$  at  $\rho > 0$ . To have a smooth origin, we therefore need to impose an additional condition on the parameters. Near  $\rho = 0$ , the relevant part of the metric is

$$ds^2 = \dots + \frac{R}{\ell_1^2} \left[ \frac{d\rho^2}{(q_1 q_2 + q_1 q_3 + q_2 q_3 + q_1 q_2 q_3)(g^2 \ell_1^2 - 1)} + \frac{(q_1 q_2 + q_1 q_3 + q_2 q_3 + q_1 q_2 q_3)}{4q_1 q_2 q_3} \rho^2 d\psi^2 \right]. \tag{3.17}$$

Thus, since  $\Delta\psi = 4\pi/k$ , the regularity condition is

$$\frac{(q_1 q_2 + q_1 q_3 + q_2 q_3 + q_1 q_2 q_3)^2 (g^2 \ell_1^2 - 1)}{q_1 q_2 q_3} = k^2. \tag{3.18}$$

When (3.16) and (3.18) are satisfied, the metric is smooth. As in the equal charge case, the surfaces of constant  $\rho$  are timelike, so there is no event horizon, and  $g^{tt} < 0$ , so  $t$  is a global time function, and there are no closed timelike curves. We have not checked whether any of these geometries have ergoregions, but we do not expect that they would.

Finally, we have to require that we can set  $A_\psi^i = 0$  at  $\rho = 0$  by an allowed gauge transformation. In the gauge used above,

$$A_\psi^i(\rho = 0) = \frac{\ell_1}{2} \left( \frac{q_j q_k (1 + q_i)}{q_i} \right)^{1/2}. \tag{3.19}$$

These gauge fields come from cross terms  $(d\phi_i + gA^i)^2$  in a Kaluza-Klein reduction from ten dimensions [24] (note there is a small difference in normalisation compared to the previous section), so the allowed large gauge transformations are  $A^i \rightarrow A^i + km^i d\psi/2g$ . The regularity condition is therefore

$$\frac{q_j q_k (1 + q_i)}{q_i} = \frac{m_i^2 k^2}{g^2 \ell_1^2} \tag{3.20}$$

where  $m_i$  are integers. The family of solitons for given asymptotic conditions are then specified by the three integers  $m_i$ . We have not attempted to solve (3.16), (3.18), (3.20) explicitly to determine the parameters in terms of these integers.

The discussion will simplify if we consider supersymmetric solutions. Translating from the parametrisation in [4] to our parametrisation, the BPS limit referred to as case A in [4]

is reached by taking  $\ell_1 \rightarrow \infty$ . To have finite gauge charges,  $q_i \ell_1^2$  must remain finite in the limit; let us define  $\bar{q}_i = q_i \ell_1^2$ .  $\ell_2$  is also fixed in this limit by (3.16). Then the regularity conditions for the BPS case reduce to

$$\ell_2^6 = \bar{q}_1 \bar{q}_2 \bar{q}_3, \tag{3.21}$$

$$\frac{g^2 (\bar{q}_1 \bar{q}_2 + \bar{q}_1 \bar{q}_3 + \bar{q}_2 \bar{q}_3)^2}{\bar{q}_1 \bar{q}_2 \bar{q}_3} = k^2, \tag{3.22}$$

and

$$\frac{\bar{q}_j \bar{q}_k}{\bar{q}_i} = \frac{m_i^2 k^2}{g^2}. \tag{3.23}$$

The first condition (3.21) corresponds to the ‘critical rotation’ condition  $\alpha^2 = q_1 q_2 q_3$  in [4], and (3.22) corresponds to equation (3.33) in [4]. The additional condition arising from imposing regularity of the gauge field (3.23) was not considered previously.

We can rewrite (3.22) as

$$g^2 \left( \sqrt{\frac{\bar{q}_1 \bar{q}_2}{\bar{q}_3}} + \sqrt{\frac{\bar{q}_2 \bar{q}_3}{\bar{q}_1}} + \sqrt{\frac{\bar{q}_1 \bar{q}_3}{\bar{q}_2}} \right)^2 = k^2. \tag{3.24}$$

Substituting (3.23) into this equation, we find

$$|m_1| + |m_2| + |m_3| = 1, \tag{3.25}$$

which cannot be satisfied, as the  $m_i$  are all nonzero integers. Note that setting some  $m_i = 0$  would correspond to setting some of the gauge charges to zero, which will not produce a regular solution, as it would imply  $\bar{H}_j(\rho = 0) = 0$ , and one of the scalar fields would blow up at the origin. Thus, as in the equal-charge case, there are no supersymmetric cases which satisfy all the regularity conditions.

#### 4. Equal charges and unequal angular momenta

All the solutions we have discussed so far have the two angular momenta equal. Recently, a solution with unequal angular momentum parameters was constructed [5, 6] (with equal charges; the most general case, unequal angular momenta and unequal charges, has not yet been found). Are there more smooth solitons as special cases of this solution? It would seem unlikely that this is possible, since for a general choice of angular momenta, there is no  $S^1$  in the  $S^3$  whose size is independent of  $\theta$  at all  $r$ . Thus, it would seem difficult to choose parameters so that we get a smooth origin for all  $\theta$ . Remarkably, this can be achieved, and there are smooth solitons.

For this case, the metric and gauge fields are

$$ds^2 = -\frac{\Delta_\theta [(1 + g^2 r^2) \rho^2 dt + 2q\nu] dt}{\Xi_a \Xi_b \rho^2} + \frac{2q\nu\omega}{\rho^2} + \frac{f}{\rho^4} \left( \frac{\Delta_\theta dt}{\Xi_a \Xi_b} - \omega \right)^2 + \frac{\rho^2 r^2 dr^2}{\tilde{\Delta}_r} + \frac{\rho^2 d\theta^2}{\Delta_\theta} + \frac{r^2 + a^2}{\Xi_a} \sin^2 \theta d\phi^2 + \frac{r^2 + b^2}{\Xi_b} \cos^2 \theta d\psi^2, \tag{4.1}$$

and

$$A = \frac{\sqrt{3}q}{\rho^2} \left( \frac{\Delta_\theta dt}{\Xi_a \Xi_b} - \omega \right), \quad (4.2)$$

where

$$\nu = b \sin^2 \theta d\phi + a \cos^2 \theta d\psi, \quad (4.3)$$

$$\omega = a \sin^2 \theta \frac{d\phi}{\Xi_a} + b \cos^2 \theta \frac{d\psi}{\Xi_b}, \quad (4.4)$$

$$\Delta_\theta = 1 - a^2 g^2 \cos^2 \theta - b^2 g^2 \sin^2 \theta, \quad (4.5)$$

$$\tilde{\Delta}_r = (r^2 + a^2)(r^2 + b^2)(1 + g^2 r^2) + q^2 + 2abq - 2Mr^2, \quad (4.6)$$

$$f = (2M + 2abqg^2) \rho^2 - q^2, \quad (4.7)$$

$$\rho^2 = r^2 + a^2 \cos^2 \theta + b^2 \sin^2 \theta, \quad (4.8)$$

$$\Xi_a = 1 - a^2 g^2, \quad \Xi_b = 1 - b^2 g^2. \quad (4.9)$$

The determinant of the metric is

$$\det g = -\frac{\rho^4 r^2 \sin^2 \theta \cos^2 \theta}{\Xi_a^2 \Xi_b^2}. \quad (4.10)$$

If  $a = b$ , this reduces to the solution in section 2, after performing the coordinate transformation and redefinition of parameters

$$\bar{r}^2 = \frac{r^2 + a^2}{\Xi_a}, \quad \bar{j} = a, \quad \bar{q} = \frac{q}{\Xi_a^2}, \quad p = \frac{M + q}{\Xi_a^3}. \quad (4.11)$$

We want to find the conditions under which the solution has a smooth origin at some value of  $r$ , where the orbits of a Killing vector

$$\xi = m\partial_\phi + n\partial_\psi \quad (4.12)$$

go smoothly to zero size. We will state the conditions without giving the sometimes tedious details of their derivation. We can only satisfy the conditions at a negative value of  $r^2$ ; we anticipate this by writing the origin as  $r^2 = -r_0^2$ . We must take  $m$  and  $n$  to be integers, so that these orbits are closed circles, and assume  $mn \neq 0$ , so the orbits do not have zero size at some point on the  $S^3$  at generic  $r$ . In this section, we will only consider the asymptotically AdS case, corresponding to  $k = 1$  in the previous sections. We therefore take  $m$  and  $n$  to be relatively prime. We can then define another Killing vector

$$\chi = k\partial_\phi + l\partial_\psi \quad (4.13)$$

with  $k$  and  $l$  integers satisfying  $ml - nk = 1$ , so that  $\xi$  and  $\chi$  form a basis for the periodic identifications equivalent to the original  $\partial_\phi, \partial_\psi$  basis. That is, in terms of new angular coordinates  $\tilde{\phi}, \tilde{\psi}$  such that  $\xi = \partial_{\tilde{\phi}}, \chi = \partial_{\tilde{\psi}}$ , the identifications are  $\tilde{\phi} \sim \tilde{\phi} + 2\pi, \tilde{\psi} \sim \tilde{\psi} + 2\pi$ .

We will need to be able to choose a gauge so that  $A \cdot \xi = 0$  at the origin. A minimal requirement is that  $A \cdot \xi$  be independent of  $\theta$  at  $r^2 = -r_0^2$ . This requires

$$\frac{n}{m} = \frac{a(a^2 - r_0^2)\Xi_b}{b(b^2 - r_0^2)\Xi_a}. \quad (4.14)$$

A smooth origin requires  $\xi = 0$  at  $r^2 = -r_0^2$ , which can be decomposed as  $\xi \cdot \xi = 0$ ,  $\xi \cdot \partial_t = 0$  and  $\xi \cdot \chi = 0$ . We also require  $\tilde{\Delta}_r(r^2 = -r_0^2) = 0$ . These conditions are all satisfied if

$$(a^2 - r_0^2)(b^2 - r_0^2) + abq = 0 \tag{4.15}$$

and

$$q(a^2 + b^2 - r_0^2 + a^2b^2g^2) + 2abM = 0. \tag{4.16}$$

It is really remarkable that all these conditions are satisfied by imposing just two constraints on the parameters: each of the three conditions on  $\xi$  involves a function of both  $r$  and  $\theta$ . These conditions ensure that at  $r = r_0$ , an  $S^1$  goes to zero size. This will be a smooth origin if

$$\frac{m^2r_0^2\delta^2}{\Xi_a^2a^2b^4(b^2 - r_0^2)^2} = 1, \tag{4.17}$$

where

$$\delta \equiv r_0^4 - 2r_0^2(a^2 + b^2 - a^2b^2g^2) + a^4 + b^4 + a^2b^2 - a^2b^2g^2(a^2 + b^2). \tag{4.18}$$

To satisfy this condition, we must have  $r_0^2 > 0$ , as stated earlier. Returning to the gauge field, a globally well-defined gauge transformation can be made to set  $A \cdot \xi = 0$  at  $r = r_0$  if

$$\frac{qam}{\Xi_a(b^2 - r_0^2)} = \frac{p}{g} \tag{4.19}$$

for some integer  $p$ . As before, from the ten-dimensional point of view,  $p$  specifies the twisting of the degenerating circle over the  $S^5$ . The five-dimensional spacetime will have a well-defined spin structure if  $m + n$  is odd, and the ten-dimensional spacetime will if  $m + n + p$  is odd.

These five conditions then fix the parameters  $r_0, a, b, q, M$  of the general solution in terms of three integers  $m, n, p$ . More explicitly,  $r_0, a$  and  $b$  are given by solving

$$m = \frac{\Xi_a ab^2(b^2 - r_0^2)}{r_0\delta}, \quad n = \frac{\Xi_b a^2b(a^2 - r_0^2)}{r_0\delta}, \tag{4.20}$$

and

$$p = \frac{abg(a^2 - r_0^2)(b^2 - r_0^2)}{r_0\delta}. \tag{4.21}$$

$M$  and  $q$  will then be given by

$$M = \frac{(a^2 - r_0^2)(b^2 - r_0^2)(a^2 + b^2 - r_0^2 + a^2b^2g^2)}{2a^2b^2}, \tag{4.22}$$

$$q = -\frac{(a^2 - r_0^2)(b^2 - r_0^2)}{ab}. \tag{4.23}$$

We should check what further restrictions on the parameters are imposed by general regularity requirements. The metric must be regular for  $r^2 > -r_0^2$ , which requires  $\tilde{\Delta}_r > 0$ ,  $\rho^2 > 0$ , and  $\Delta_\theta > 0$  for all  $r^2 > -r_0^2$ . Requiring  $\rho^2 > 0$  implies

$$a^2 - r_0^2 > 0, \quad b^2 - r_0^2 > 0. \tag{4.24}$$

Requiring  $\Delta_\theta > 0$  implies

$$\Xi_a > 0, \quad \Xi_b > 0. \quad (4.25)$$

Requiring that  $-r_0^2$  is the largest root of  $\tilde{\Delta}_r$  implies  $\delta > 0$ . There is no obvious inconsistency between these inequalities and (4.20), (4.21), so we imagine that for generic values of the integers, we can choose a solution of (4.20), (4.21) which satisfies these conditions. We note that they automatically imply that  $M > 0$ .

Requiring  $\tilde{\Delta}_r > 0$  also ensures that there are no horizons in the spacetime, as it ensures that the surfaces of constant  $r$  are timelike for  $r^2 > -r_0^2$ . To show that the spacetime does not contain closed timelike curves, we calculate

$$g^{tt} = -\frac{(r^2 + r_0^2)}{\tilde{\Delta}_r \Delta_\theta \rho^2 a^2 b^2} \{a^2 b^2 \Xi_a \Xi_b [(r^2 + r_0^2)^2 + (\rho_0^2 - 2r_0^2 + a^2 + b^2)(r^2 + r_0^2)] + \delta[\Xi_b a^2 (b^2 - r_0^2) \sin^2 \theta + \Xi_a b^2 (a^2 - r_0^2) \cos^2 \theta]\}, \quad (4.26)$$

where we have used (4.22), (4.23) in simplifying the form of  $g^{tt}$ , and  $\rho_0^2 \equiv \rho^2(r^2 = -r_0^2)$ . Thus,  $g^{tt} < 0$  for all  $r^2 \geq -r_0^2$  (recall that there is an  $r^2 + r_0^2$  factor in  $\tilde{\Delta}_r$ , so  $g^{tt}$  is strictly less than zero at  $r^2 = -r_0^2$ ). That is,  $dt$  is an everywhere timelike one-form, and  $t$  is a global time function. These solitons are stably causal and hence free of closed timelike curves. Again, we have not attempted to check for ergoregions, which would be a tedious exercise in this case.

Finally, we consider the geometry of the surface at  $r^2 = -r_0^2$ , with coordinates  $t, \theta, \tilde{\psi}$ , where  $\tilde{\psi}$  is the  $2\pi$  periodic coordinate associated with  $\chi = k\partial_\phi + l\partial_\psi$ . At  $r^2 = -r_0^2$ ,

$$\chi \cdot \chi = \frac{r_0^2 \delta^2 \sin^2 \theta \cos^2 \theta}{a^4 b^4 \rho_0^4 \Xi_a^2 \Xi_b^2} [a^2 (b^2 - r_0^2) \Xi_b \sin^2 \theta + b^2 (a^2 - r_0^2) \Xi_a \cos^2 \theta], \quad (4.27)$$

where we have used (4.22), (4.23) and  $lm - kn = 1$ . This circle degenerates at  $\theta = 0$  and at  $\theta = \pi/2$ . At  $\theta = 0$ ,  $\rho_0^2 = a^2 - r_0^2$  and  $\Delta_\theta = \Xi_a$ , so

$$ds^2 \approx \dots + \frac{a^2 - r_0^2}{\Xi_a} \left( d\theta^2 + \sin^2 \theta \frac{d\tilde{\psi}^2}{n^2} \right). \quad (4.28)$$

Similarly, at  $\theta = \pi/2$ ,  $\rho_0^2 = b^2 - r_0^2$  and  $\Delta_\theta = \Xi_b$ , so

$$ds^2 \approx \dots + \frac{b^2 - r_0^2}{\Xi_b} \left( d\theta^2 + \sin^2 \theta \frac{d\tilde{\psi}^2}{m^2} \right). \quad (4.29)$$

The geometry will therefore have a  $\mathbb{Z}_{|n|}$  orbifold singularity at  $r^2 = -r_0^2, \theta = 0$  and a  $\mathbb{Z}_{|m|}$  orbifold singularity at  $r^2 = -r_0^2, \theta = \pi/2$ .<sup>6</sup> That is, the only truly smooth case is  $m^2 = n^2 = 1$ .

Let us consider in particular the supersymmetric case. The BPS condition for these solutions is [6]

$$\frac{M}{q} = 1 + (a + b)g. \quad (4.30)$$

---

<sup>6</sup>These orbifold singularities appear only in the surface at  $r^2 = -r_0^2$ . At other values of  $r$ , the  $\phi$  circle shrinks as  $\theta \rightarrow 0$  and  $\psi$  circle shrinks as  $\theta \rightarrow \pi/2$ , and these are both smooth origins.

Using (4.22), (4.23), this can be rewritten as

$$r_0^2 = (a + b + abg)^2, \tag{4.31}$$

and (4.22) becomes

$$M = -(1 + ag + bg)(1 + ag)(1 + bg)(2a + b + abg)(2b + a + abg), \tag{4.32}$$

as in [6]. The integer conditions (4.20), (4.21) are then

$$m = -\frac{(1 - ag)(2b + a + abg)}{(a + b + abg)(3 + 5ag + 5bg + 3abg^2)}, \tag{4.33}$$

$$n = -\frac{(1 - bg)(2a + b + abg)}{(a + b + abg)(3 + 5ag + 5bg + 3abg^2)}, \tag{4.34}$$

and

$$p = -\frac{g(2b + a + abg)(2a + b + abg)}{(a + b + abg)(3 + 5ag + 5bg + 3abg^2)}. \tag{4.35}$$

There are thus three conditions on the two unknowns  $a, b$ . Although we have not made a systematic search, it seems unlikely that there will be any supersymmetric solitons. In the special case  $m^2 = n^2 = 1$ , when the geometry is completely smooth, the first two equations reduce to  $a^2 = b^2$ , so the analysis of section 2 shows that there are no supersymmetric solitons with  $m^2 = n^2 = 1$ . In [6], (4.33) was imposed with  $m = -1$ , but the other conditions were not. To obtain a smooth metric, we must also impose (4.34) for some value of  $n$ , and to make the gauge potential well-defined we must also satisfy (4.35).

## 5. Asymptotically Poincaré solutions

So far, we have considered solitons which asymptotically approach AdS in global coordinates. However, it is also possible to consider solitons which asymptotically approach AdS in Poincaré coordinates with some direction compactified, as in the AdS soliton of [19]. We will see that a large mass limit of our global AdS solitons gives analogues of the AdS soliton with non-zero gauge fields. We can exploit the twisting on the sphere to allow periodic boundary conditions for the fermions on the compact circle in Poincaré coordinates. However, we again fail to find any supersymmetric solitons.

We analyse this for the case of unequal charges and equal angular momenta.<sup>7</sup> Consider the solutions (3.10), (3.15) and make the scaling<sup>8</sup>

$$\ell_1 = \gamma \bar{\ell}_1, \quad \ell_2 = \gamma \bar{\ell}_2, \quad q_i \text{ fixed}, \tag{5.1}$$

---

<sup>7</sup>We do not see any easy way to extend this scaling argument to the metric with equal charges and unequal angular momenta considered in the previous section. For an asymptotically Poincaré solution, we only expect to need one parameter for the momentum along the flat directions, so it may be that the solution obtained here is sufficiently general.

<sup>8</sup>Note that this scaling is different from the one used in e.g. [25, 26] to obtain toroidal black holes. This scaling is dictated by the regularity conditions (3.16), (3.18).

and scale the coordinates as

$$\rho = \gamma \bar{\rho}, \quad t = \frac{\bar{t}}{\gamma}, \quad \theta = \frac{2x}{\gamma}, \quad \psi + \phi = \frac{2z}{\gamma}. \quad (5.2)$$

Then when we take  $\gamma \rightarrow \infty$ , we obtain a solution

$$ds^2 = -Rg^2 d\bar{t}^2 + \frac{R\bar{\rho}^2}{g^2 f_1} d\bar{\rho}^2 + R(dx^2 + x^2 d\phi^2) + \frac{f_1}{R^2} dz^2, \quad (5.3)$$

with

$$f_1 = R^3 + \frac{\bar{\ell}_2^6}{\bar{\ell}_1^2} \bar{\rho}^2 - \bar{\ell}_2^6, \quad (5.4)$$

and  $R^3 = \bar{H}_1 \bar{H}_2 \bar{H}_3$ , where

$$\bar{H}_i = \bar{\rho}^2 + \bar{\ell}_1^2 q_i. \quad (5.5)$$

The gauge field becomes

$$A^i = \sqrt{q_i + 1} \frac{\bar{\ell}_2^3}{(\bar{\rho}^2 + \bar{\ell}_1^2 q_i)} dz, \quad (5.6)$$

and the scalars are

$$e^{\frac{2}{\sqrt{6}}\varphi_1} = X_3, \quad e^{\sqrt{2}\varphi_2} = \frac{X_2}{X_1}, \quad X_i = \frac{R}{\bar{H}_i}. \quad (5.7)$$

Note that the form of the metric simplifies because  $Y \approx g^2 f_1$  in the limit and the  $f_2 dt/f_1$  term becomes negligible. This family of geometries can also be obtained by a double analytic continuation of charged toroidal black holes [27, 28].

We would like to see what solitons can be obtained from this family. The limiting procedure gave us the solution (5.3) with  $z \in (-\infty, \infty)$ , but to obtain smooth solitons we must make periodic identifications  $z \sim z + \Delta z$ . This solution appears to depend on five parameters,  $\bar{\ell}_1$ ,  $\bar{\ell}_2$ , and  $q_i$ . However, two of these parameters are coordinate degrees of freedom. This can be made manifest by making the coordinate transformation

$$r^2 = \frac{\bar{\ell}_1}{\bar{\ell}_2^3} (\bar{\rho}^2 - \bar{\ell}_1^2), \quad \tau = \frac{\bar{t} \bar{\ell}_2^{3/2}}{\bar{\ell}_1^{1/2}}, \quad X = \frac{x \bar{\ell}_2^{3/2}}{\bar{\ell}_1^{1/2}}, \quad Z = \frac{z \bar{\ell}_2^{3/2}}{\bar{\ell}_1^{1/2}} \quad (5.8)$$

so that

$$ds^2 = -Rg^2 d\tau^2 + \frac{Rr^2}{g^2 f_1} dr^2 + R(dX^2 + X^2 d\phi^2) + \frac{f_1}{R^2} dZ^2, \quad (5.9)$$

with

$$f_1 = R^3 + r^2 \quad (5.10)$$

and

$$\bar{H}_i = r^2 + \frac{\bar{\ell}_1^3}{\bar{\ell}_2^3} (q_i + 1). \quad (5.11)$$

The metric in this coordinate system depends only on the combinations  $\frac{\bar{\ell}_1^3}{\bar{\ell}_2^3} (q_i + 1)$ , so there are really only three independent parameters. Rather than work in this new coordinate



system, it is more convenient to fix this gauge freedom by choosing  $\bar{\ell}_2^6 = \bar{\ell}_1^6 q_1 q_2 q_3$ , and fixing the period of  $\Delta z$ . We reiterate that for these asymptotically Poincaré solutions, this is a choice of gauge, not a regularity condition.

Let us now consider the regularity conditions, working with the radial coordinate  $\bar{\rho}^2$ . To have a regular origin at  $\bar{\rho}^2 = \bar{\rho}_0^2$  in the geometry (5.3) only requires that  $f_1(\bar{\rho}_0^2) = 0$  and that the circle have the correct proper radius. With the choice  $\bar{\ell}_2^6 = \bar{\ell}_1^6 q_1 q_2 q_3$ ,

$$f_1 = \bar{\rho}^6 + \bar{\rho}^4 \bar{\ell}_1^2 (q_1 + q_2 + q_3) + \bar{\rho}^2 \bar{\ell}_1^4 (q_1 q_2 + q_1 q_3 + q_2 q_3 + q_1 q_2 q_3), \quad (5.12)$$

so there is a root of  $f_1$  at  $\bar{\rho}^2 = 0$ . If  $\bar{\ell}_1^2 q_i > 0$ ,  $\bar{H}_i(\bar{\rho}^2 = 0) > 0$ , so there is no singularity at larger  $\bar{\rho}$ , and this will be the largest root of  $f_1$ . We use the coordinate freedom to set  $\Delta z = 2\pi$  to make this look as much as possible like the global AdS case: the condition to have a regular origin is then

$$\frac{(q_1 q_2 + q_1 q_3 + q_2 q_3 + q_1 q_2 q_3)^2 g^2 \bar{\ell}_1^2}{q_1 q_2 q_3} = 1. \quad (5.13)$$

The regularity conditions for the gauge fields are

$$\frac{q_j q_k (1 + q_i)}{q_i} = \frac{m_i^2}{g^2 \bar{\ell}_1^2}. \quad (5.14)$$

We see that we obtain a very similar family of smooth solitons to the ones we had in the global AdS case. There is a discrete family of geometries labelled by the integers  $m_i$ , none of which are supersymmetric.

These asymptotically Poincaré solitons will arise as the near-core limit of asymptotically flat solitons carrying D3-brane charge, just as global  $\text{AdS}_3 \times S^3$  arises as the near-core limit of the asymptotically flat six-dimensional solitons in [1, 2]. The relevant asymptotically flat solutions can be constructed by double analytic continuation of rotating black D3-brane solutions.

From the ten-dimensional perspective, the circle which shrinks at  $\bar{\rho}^2 = 0$  is  $z$  twisted  $m_i$  times over the three  $U(1)$  angular directions in the  $S^5$ . Thus, if  $m_1 + m_2 + m_3$  is odd, the ten-dimensional soliton has a spin structure where the fermions are periodic around the  $z$  direction at fixed position on the sphere. The absence of a supersymmetric soliton in this case may somehow be connected with the need to pick out one of the flat directions to play the role of the degenerating circle.

## 6. Asymptotically flat limit

We have found non-supersymmetric asymptotically AdS solitons in five-dimensional gauged supergravity. We would like to understand the relation to the asymptotically flat solitons in [7]: the three-charge metrics in that paper can be Kaluza-Klein reduced to give asymptotically flat solutions in five-dimensional ungauged supergravity, and one might expect that they would correspond to a  $g \rightarrow 0$  limit of the global AdS solutions. Comparing the qualitative properties of the two families of solitons, we see that there are some important differences between the two cases: in the present paper, we have found solitons for equal

or unequal angular momenta, while the asymptotically flat solitons always have unequal angular momenta. The asymptotically AdS solitons are labelled by integers which specify all the parameters including the charges, whereas in the asymptotically flat case, we were free to choose the charges, and for each choice of charges, there was a family of solitons labelled by integers. The asymptotically AdS solitons do not have ergoregions, while the asymptotically flat solitons do.

Despite these differences, we can make a close contact between the limit as  $g \rightarrow 0$  of the analysis in section 4 and the analysis in [7]. However, the asymptotically flat solitons will be a distinct family which only exist when  $g = 0$ .

We cannot take such a limit for the solitons with equal angular momentum, as the condition  $g^2 \ell_1^2 > 1$  would require us to take some of the parameters to infinity. There is no similar condition in the case with unequal angular momenta, however, and if we take the limit  $g \rightarrow 0$  for the metric (4.1), we obtain an asymptotically flat solution, which for ease of comparison to [7] we rewrite as

$$\begin{aligned}
 ds^2 = & -\frac{F}{\rho^4} \left[ dt + \frac{(2Ma + qb)\rho^2 - q^2a}{F} \sin^2 \theta d\phi + \frac{(2Mb + qa)\rho^2 - q^2b}{F} \cos^2 \theta d\psi \right]^2 \\
 & + \rho^2 \left\{ \frac{r^2 dr^2}{\tilde{\Delta}_r} + d\theta^2 + \sin^2 \theta d\phi^2 + \cos^2 \theta d\psi^2 \right. \\
 & + \frac{1}{F} \left[ ((a^2 - b^2)\rho^2 + (2Mb^2 + 2qab)) \sin^4 \theta d\phi^2 \right. \\
 & + (2Mab + q(a^2 + b^2)) \sin^2 \theta \cos^2 \theta d\phi d\psi \\
 & \left. \left. + ((b^2 - a^2)\rho^2 + (2Ma^2 + 2qab)) \cos^4 \theta d\psi^2 \right] \right\}, \tag{6.1}
 \end{aligned}$$

where  $\rho^2$  is given in (4.8),  $F \equiv \rho^4 - 2M\rho^2 + q^2$ , and once we have set  $g = 0$ ,

$$\tilde{\Delta}_r = (r^2 + a^2)(r^2 + b^2) + q^2 + 2abq - 2Mr^2. \tag{6.2}$$

This is the same as the five-dimensional form of the general solution considered in [7] (for all three charges equal). To bring it to the form used in [7], we should redefine the parameters by

$$2M = \bar{M}(\cosh^2 \delta + \sinh^2 \delta), \quad q = -\bar{M} \sinh \delta \cosh \delta, \tag{6.3}$$

$$a = a_1 \sinh \delta - a_2 \cosh \delta, \quad b = a_1 \cosh \delta - a_2 \sinh \delta, \tag{6.4}$$

and define a new radial coordinate

$$\bar{r}^2 = r^2 + (a_1^2 + a_2^2) \sinh^2 \delta - 2a_1 a_2 \sinh \delta \cosh \delta - \bar{M} \sinh^2 \delta. \tag{6.5}$$

To understand the relation between the asymptotically AdS and asymptotically flat solitons, we would like to re-express the conditions (4.15), (4.16) and the integers (4.20), (4.21) in terms of these new parameters. We can think of (4.16) as determining  $r_0^2$ ; in terms of the new parameters, it gives

$$r_0^2 = \frac{a_1 a_2}{\sinh \delta \cosh \delta}. \tag{6.6}$$

Substituting this into (4.15) gives

$$\bar{M} = a_1^2 + a_2^2 - a_1 a_2 \frac{\cosh^6 \delta + \sinh^6 \delta}{\cosh^3 \delta \sinh^3 \delta}, \quad (6.7)$$

which matches (3.15) in [7]. The expressions for the integers  $m$  and  $n$  in (4.20) become

$$m = \frac{(\sinh \delta \cosh \delta)^{3/2} (a_1 \cosh^3 \delta - a_2 \sinh^3 \delta)}{\sqrt{a_1 a_2} (\cosh^6 \delta - \sinh^6 \delta)}, \quad (6.8)$$

$$n = \frac{(\sinh \delta \cosh \delta)^{3/2} (a_1 \sinh^3 \delta - a_2 \cosh^3 \delta)}{\sqrt{a_1 a_2} (\cosh^6 \delta - \sinh^6 \delta)}. \quad (6.9)$$

These are thus the same as the integers  $m$  and  $n$  labelling the solitons in [7]. When we set  $g = 0$  in (4.21), on the other hand, it simply reduces to

$$p = 0. \quad (6.10)$$

This is an allowed value for  $p$ , from the point of view of the integer quantisation. Thus, we recover the asymptotically flat solitons of [7] as the special case  $g = 0$ ,  $p = 0$  of the solutions in the previous section.<sup>9</sup> Because (4.21) is automatically satisfied, we have one free parameter, which we can use to choose the charge arbitrarily.

Note that  $p \propto mng$ , so for  $g \neq 0$  we cannot set  $p = 0$ , as it would contradict our assumption that  $mn \neq 0$ . Thus, the asymptotically flat solitons are a distinct family which appear when  $g = 0$ , and not the special case  $g = 0$  of the asymptotically AdS solitons. This explains why the properties of the asymptotically AdS and asymptotically flat cases are different.

In addition, while there is a close relation between the two families as solutions in five dimensions, their interpretation in string theory is quite different: the gauged supergravity solutions arise from type IIB compactified on  $S^5$ , with a nonzero five-form field strength proportional to  $g$ , and angular momenta on the  $S^5$  proportional to  $q$ . The asymptotically flat solutions arise for example from type IIB compactified on  $T^5$ , with a three-form field strength proportional to  $q$  and a momentum proportional to  $q$  on one of the  $T^5$  directions.

## 7. Conclusions

In this paper, we have found non-supersymmetric solitons in five-dimensional gauged supergravity. Surprisingly, we were unable to find any supersymmetric solitons. The solutions have a circle which degenerates smoothly in the interior of the spacetime. As in previous studies of asymptotically flat and  $\text{AdS}_3 \times S^3$  cases, a twisting plays a crucial role: the circle which degenerates in the interior is twisted over the  $S^5$  in the ten-dimensional metric. These non-supersymmetric solitons have a different character to the 1/2 BPS asymptotically  $\text{AdS}_5 \times S^5$  geometries found in [3]: In particular, they have a non-contractible  $S^2$ , so they have a different topology.

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<sup>9</sup>As noted above, the five-dimensional geometry has orbifold singularities at  $r^2 = -r_0^2$ ,  $\theta = 0, \pi/2$  for general values of  $m, n$ . The asymptotically flat geometry is completely smooth only in six dimensions.

There are a number of directions in which this work could be extended. These solitons will have corresponding states in the dual  $\mathcal{N} = 4$  super Yang-Mills theory [29, 30]. One can read off the charges of these states and the expectation values of field theory operators from the bulk solutions given above, but explicitly identifying the corresponding field theory states in the absence of supersymmetry will be challenging, and we currently have no clear idea how to proceed.<sup>10</sup> Nonetheless, this is an important problem to address: it is hard to imagine we will get much simpler examples of non-supersymmetric solitons to work with, and extending the AdS/CFT dictionary beyond the supersymmetric context is important for a host of reasons.

It would also be interesting to compare the properties of these soliton geometries to those of the charged rotating black holes, to explore if the solitons could be interpreted as microstates of the black holes following the ideas of [22]. Another sense in which solitons and black holes could be compared is, following [31, 32], to ask if there are black holes with a winding tachyon whose condensation could lead to one of these solitons.

Their stability should also be investigated. We showed in section 2 that the solitons with equal charges and equal angular momenta had no ergoregions. We did not explore the existence of ergoregions for the other cases; it was enough to observe that we had some examples in the asymptotically AdS case of non-supersymmetric soliton solutions without ergoregions (unlike in the asymptotically flat case). Hence, these solutions can avoid the instability argument of [33]. This of course does not imply that they are stable; further study is necessary.

We will finally remark on the extension of this analysis to different numbers of dimensions. Similar families of charged rotating black hole solutions have been found for the gauged supergravity theories in four and seven dimensions [34–37], and we would like to find solitons in these theories as well. It is clear that in the four-dimensional case, it will not be possible to construct smooth solitons in the same way. There is only one  $U(1)$  direction in the transverse two-sphere in the four-dimensional solution, and it shrinks to zero size at the north and south poles of the two-sphere. Thus, its size is always a function of  $\theta$ , and it will not be possible to combine it with the radial coordinate to form a smooth origin for all  $\theta$ .

The seven-dimensional case seems at first blush more promising. The spacetime involves an  $S^5$ , and the metric is naturally written in coordinates that write the  $S^5$  as an  $S^1$  bundle over  $\mathbb{C}P^2$  [37],

$$ds^2 = \dots + A \left( d\xi^2 + \frac{1}{4} \sin^2 \xi (\sigma_1^2 + \sigma_2^2) + \frac{1}{4} \sin^2 \xi \cos^2 \xi \sigma_3^2 \right) + B(\sigma + C dt)^2, \quad (7.1)$$

where the  $\sigma_i$  are the left-invariant one-forms on  $S^3$  (2.6-2.8), and

$$\sigma = d\tau + \frac{1}{2} \sin^2 \xi \sigma_3. \quad (7.2)$$

It therefore seems natural to look for smooth solitons where the radial direction combines with the  $\tau$  circle to form a smooth origin. Supersymmetric solutions with smooth metrics

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<sup>10</sup>It may be possible to relate the twisting of the degenerating circle over the  $S^5$  to something like spectral flow in the dual field theory, but this is probably not the whole story.

were indeed found in [4]. However, there is a problem with the gauge fields. In the charged black holes of [37], the solutions involve a non-trivial three-form potential

$$A_{(3)} = \frac{2mas^2}{\Xi\Xi_-(r^2+a^2)}\sigma \wedge J, \quad (7.3)$$

where  $J$  is the Kähler form on  $\mathbb{CP}^2$ . If we consider the holonomy of this three-form obtained by integrating it over the  $\tau$  circle and an  $S^2 \subset \mathbb{CP}^2$  parametrised by  $\theta, \phi$ ,

$$\oint_{S^1 \times S^2} A_{(3)} = \frac{2mas^2}{\Xi\Xi_-(r^2+a^2)} 2\pi^2 \sin^2 \xi. \quad (7.4)$$

Since this holonomy depends on the coordinate  $\xi$  labelling the two-sphere we choose, it is impossible to choose parameters such that this holonomy is trivial for all the possible cycles. (Setting  $mas^2 = 0$  would set either the charge or the angular momentum to zero, and there are no solutions with smooth metrics for these cases.) Thus, we cannot find any solitons with smooth three-form in this family of solutions.

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